The place of thinking in the (mathematics) curriculum

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Thinking is assumed to be an aim of education but only recently has it been specifically mentioned in curriculum documents. Even now it is often couched in general terms and I believe that what we mean by thinking needs to be ‘unpacked’. However, unpacking thinking raises issues about how we might structure curriculum documents so that thinking is embedded alongside the content and other important aspects of the curriculum. In focussing on this issue my intention is not to provide answers but rather to begin and stimulate the debate.

Introduction

When most people consider education they assume that ‘thinking’ is involved, but it is surprising how rarely it has been specifically mentioned in curriculum documents over the last four decades. Two questions that seem to need debate are: what do we mean by thinking both generally and in separate subjects, and, how might curriculum documents be written so that thinking is given due emphasis?

Subject-based curriculum

In the past most curriculum documents focussed on the content (facts and procedures) that students were expected to master. More recently this has been extended to content and process (knowing and doing). Now, in addition to content and process, some countries have introduced ‘key competencies’ that typically include thinking. These three aspects (knowing, doing and thinking) overlap to a considerable degree and impinge on every subject.

While I am concerned that such subject-based curriculum compartmentalize knowledge and reflect western ways of thinking about knowledge rather than more holistic ways, I do not expect the curriculum to change to an integrated one. However, although the school curriculum is likely to remain subject based there is a need to address what knowing, doing and thinking mean within each subject. I am not an expert across all subjects but I can give some examples from my area of interest which is mathematics, and I assume that other subject specialists can consider their subjects in similar ways.

Knowing, doing and thinking

The content of mathematics was traditionally subdivided into arithmetic, algebra, geometry, statistics, and so on, although more recently these have been renamed as number, patterns, space and chance. Each of these were further subdivided into broad or specific objectives involving particular facts or procedures.

Since the eighties the ‘processes’ have been developed. Typically in mathematics these included: problem solving, logical reasoning, and communicating and some people and in some countries these were extended to include using technology and making connections.
Thinking however has not usually been mentioned in the curriculum. When it has, it was usually at a general level in three forms: critical, creative, and meta-cognitive thinking. All subjects including mathematics can contribute to these but this is not enough. I am also concerned with the aspects of thinking that relate to and may be unique to each subject (including mathematics). I believe that these need discussion and need to be made explicit so that at all levels of curriculum planning they can be given due consideration.

Thinking within mathematics
Within mathematics Mason’s book (Mason, with Burton & Stacey, 1982) discussed conjecturing, justifying and convincing as thinking skills. These can be considered as thinking or as part of the processes logical reasoning and communicating; and this illustrates the difficulty of partitioning content, process and thinking.

In the past I had thought of patterning, changing representations, visualising, and understanding variation as being four important aspects of mathematical thinking. More recently I have come to agree with my colleague John Mason’s view that generalising (and specialising) are even more important. Mason (2002) emphasised this when he said that,

*a lesson without the opportunity to generalise is not a lesson in mathematics.*

Thus, without going deeply into an analysis of thinking skills, the ideal mathematics curriculum might include:
- critical thinking
- creative thinking
- meta-cognitive thinking
- generalising (and specialising)
- conjecturing
- justifying (convincing and proving)
- visualising
- changing representations
- patterning
- understanding variation.

Further, these occur alongside both the content areas (number/arithmetic, algebra, geometry, statistics, …) and the mathematical processes.

Habits of mind
Another way of thinking about thinking is to consider the ‘habits of mind’ that we want our students to develop and retain after they have forgotten the content of subjects that they learnt at school. Costa & Kallick (1999, p. xvii-xviii) provided a list of habits:
- persisting
- managing impulsivity
- listening with understanding and empathy
- thinking flexibly
- thinking about thinking (metacognition)
- striving for accuracy
- questioning and posing problems
- applying past knowledge to new situations
- thinking and communicating with clarity and precision
- gathering data through all senses
- creating, imagining, innovating
- responding with wonderment and awe
- taking responsible risks
- finding humor
- thinking independently
- remaining open to continuous learning.

Clearly there is a considerable overlap between my list for mathematics and this general one, and many mathematics teachers would think that some of these general habits need to be a focus within mathematics.

Other writers have looked at this topic differently. For example, in mathematics, Cuoco, Goldenberg, & Mark (1997) developed subject specific habits of mind as organizing principles for the curriculum. They listed:

<table>
<thead>
<tr>
<th>Students</th>
<th>Mathematicians</th>
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<tbody>
<tr>
<td>- should be pattern sniffers</td>
<td>- talk big and think small</td>
</tr>
<tr>
<td>- should be experimenters</td>
<td>- talk small and think big</td>
</tr>
<tr>
<td>- should be describers</td>
<td>- use functions</td>
</tr>
<tr>
<td>- should be tinkerers</td>
<td>- mix deduction and experiment</td>
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<tr>
<td>- should be inventors</td>
<td>- push the language</td>
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<tr>
<td>- should be visualizers</td>
<td>- use intellectual chants</td>
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<td>- should be conjecturers</td>
<td>- should be guessers</td>
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<table>
<thead>
<tr>
<th>Geometers</th>
<th>Algebraists</th>
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<tr>
<td>- use proportional reasoning</td>
<td>- like a good calculation</td>
</tr>
<tr>
<td>- use several languages at once</td>
<td>- use abstraction</td>
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<tr>
<td>- use one language for everything</td>
<td>- like algorithms</td>
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<tr>
<td>- love systems</td>
<td>- break things into parts</td>
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<tr>
<td>- worry about things that change</td>
<td>- extend things</td>
</tr>
<tr>
<td>- worry about things that don’t change</td>
<td>- represent things</td>
</tr>
<tr>
<td>- love shapes</td>
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**Analytic and synthetic thinking.**

About twenty five years ago when curriculum appeared to be in its heyday behaviourism was the accepted learning theory. Working within this paradigm it seemed reasonable to break subjects into specific ‘objectives’ and teach these in sequential form. This breaking up of subjects emphasised analytic thinking. Too often the specific objectives were taught but were not explicitly considered tied back together again (synthetic thinking).

Constructivism is now a more accepted theory for learning. Consequently we are concerned with building on students’ existing knowledge and accept that each student builds up their ‘knowledge schema’ in a unique way. In this situation one cannot assume that the analytic approach is the most appropriate.

One example of the need to begin with a synthetic rather than analytic approach is with geometry. Crowley (1987) described the Van Hiele levels and how learners need experience seeing shapes as wholes before considering their properties, and later the relationships between similar shapes. Only then can they begin to understand definitions and discriminate between definitions and properties. I would assume that similar levels exist within other content areas and this reinforces my belief that both synthetic and analytic thinking are needed, but neither is sufficient by itself.
Learning task/activity
I want to break at this stage for a few minutes so that you can spend a little time on this task that involves thinking and gives us a shared experience to reflect on.

<table>
<thead>
<tr>
<th>Task: You have a square of paper.</th>
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<tbody>
<tr>
<td>Can it be cut into exactly eleven squares?</td>
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<tr>
<td>(Being ‘cut’ and ‘exactly’ implies no scraps will remain.)</td>
</tr>
<tr>
<td>If so, how? If not, why not?</td>
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</tbody>
</table>

Reflecting on this task.
- Firstly, for me it is a mathematical task. I am sure that all of you were able to make some progress with it even if you suffer from mathematics phobia—perhaps you did not think of it as mathematics!
- Secondly, it involved critical thinking, that is, questioning assumptions. I am sure that many of you asked yourselves whether all the squares were the same size,
- Thirdly, it is not a problem with only one answer. While the first part of the answer is yes, the how part has at least 16 ways, and even then I am only counting the sizes of the squares and not the ways that the original might be cut.
- Fourthly, we can generalize by changing the number from 11? (In fact it is true for all numbers except 2, 3, and 5—but can you prove that?)
- Fifthly, is this true if we replace the word square with some other shape? (For example, equilateral triangle or cube)
- Most importantly, how might this activity/task, fit into our mathematics curriculum?

In my experience most twelve-year olds get their first answer in about half the time that teachers take, and with a little encouragement most school students are willing to carry on beyond this while adults seem to think that such a question is completed when one answer is produced. The task is rich in thinking skills though different people are likely to use different skills because they are building on different prior knowledge. However, the final question, how might this fit into a curriculum is a major concern.

Structuring the curriculum
Traditionally subject-curriculum documents have been divided into strands. This concerns me as many topics belong in more than one strand and there are important links between strands. For example in mathematics: number and algebra are used in geometry, geometry provides semi-concrete examples for some algebraic and numerical work, and number concepts can be learnt from statistics as well as statistics being taught after the basic concepts are understood.

This ‘strand’ nature of the curriculum was exacerbated within mathematics with the introduction of the mathematical processes or ‘doing’ strand. This was evident with the ‘Standards’ from the United States of America (National Council of Teachers of Mathematics, 1989), and the New Zealand mathematics curriculum (Learning Media/Ministry of Education, 1992); and I am sure, in documents from other regions. The process strand was introduced to supplement the traditional content strands. This seemed reasonable when one considered the alternative. If content (knowing) and process (doing) had been conceptualised in terms of a matrix with each aspect of each process overlapping each sub-topic of each content strand, then teachers would have been faced with an overfull curriculum and not been sure how to begin their planning.
However, with separate strands the processes were often taught separately from content rather than integrated with it. In the introductory sections of these curriculum documents the interrelatedness was discussed, but the reality is that teachers usually read the introduction once but then constantly use the strands for planning teaching.

This situation becomes even more complex when one thinks of a third aspect (thinking) being introduced. A three-dimensional matrix is one way to think of the curriculum, but that becomes even more confusing for teachers. The only solution seems to be to reconceptualize how we use curriculum documents. Of course it would be possible to have a process-based curriculum or a thinking-based curriculum, but the problem would remain the curriculum needs all three aspects.

A further level of complexity arises when one adds to curriculum by considering the aims for education (or competencies) and how each subject might contribute to these. For example, how might a subject such as school mathematics contribute to fostering aims such as these from Munro (1969):

- the urge to enquire
- concern for others
- the desire for self-respect

or to developing the Ministry of Education (2006)competencies of:

- managing self
- relating to others
- participating and contributing
- using language, symbols, and texts

or whatever other aims, visions, or values that are specified to be considered. While this additional level of complexity might make the task of restructuring curriculum overwhelming, it is important to remember that these aims are more important than specific subject knowledge.

It is possible to write curriculum guide-lines to show how knowing, doing, thinking, and fostering our aims of education might be facilitated within a subject, but each guideline may still remain focussed on only one aspect. In such a situation it may be preferable to consider how we might use curriculum documents differently.

**Using the curriculum**

Most teachers think of the curriculum documents as the basis for planning, but what is meant by planning? I have been heartened by the notion from Davis (1996, pp 272–273), who, writing within the context of teaching mathematics talks of ‘curriculum occasioning’. He suggests that the role of a teacher who truly listens to students is ‘neither telling nor orchestrating’ but involves enacting a ‘planned, but not predetermined curriculum’. This suggests planning flexibly then responding to students, but the question remains, how does one go about this.

It is interesting to remember what happened in the past. Many school mathematics teachers (and presumably other subject teachers too) followed the textbook. This was reasonable at the time as few resources often existed, some teachers lacked a comprehensive subject knowledge, and textbook writers were assumed to be experts. Now many more resources are available, pre-service teacher education is more comprehensive, and students come from more varied backgrounds, so one-size does not fit all. However, it may still be worthwhile to consider resources more centrally in
curriculum planning. One possibility seems to me to think of a range of ‘rich learning tasks/activities’ and use these in our planning. My criteria for learning tasks/activities to be ‘rich’ are modified from the work of Ahmed (1987) and Cox (1998) for another purpose (Begg, 2006), see table 1.

Table 1: **Criteria for rich learning tasks/activities**

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Criteria</th>
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<tbody>
<tr>
<td><strong>Approach</strong></td>
<td>– approach the unknown through what is known to the students</td>
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<td>– be accessible to all students at the start</td>
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<td></td>
<td>– allow further challenges and be extendible</td>
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<td></td>
<td>– challenge the better students without overwhelming the weaker ones</td>
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<tr>
<td><strong>Properties</strong></td>
<td>– be interesting to the students, and to achieve this, to the teacher</td>
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<td></td>
<td>– have an element of surprise</td>
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<td></td>
<td>– be enjoyable (that is, engaging)</td>
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<td></td>
<td>– should not trivialize the subject</td>
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<tr>
<td>** Appropriateness**</td>
<td>– introduce material within the programme at a time relative to its use</td>
</tr>
<tr>
<td><strong>Possibilities</strong></td>
<td>– provide opportunities for constant review</td>
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<td></td>
<td>– invite students to make decisions</td>
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<td></td>
<td>– involve students in speculating, hypothesis making and testing, proving or explaining, reflecting, interpreting</td>
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<td></td>
<td>– do not restrict students from searching in other directions</td>
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<td></td>
<td>– promote discussion and communication</td>
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<td></td>
<td>– encourage originality/invention</td>
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<td></td>
<td>– encourage ‘what if’ and ‘what if not’ question</td>
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<tr>
<td><strong>Focus</strong></td>
<td>– emphasize key general principles more than technical details</td>
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<td></td>
<td>– provide specific illustrations of general principles</td>
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<td></td>
<td>– be seen both as an end and as a basis for subsequent work and study</td>
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<td></td>
<td>– avoid the temptation to teach too much material</td>
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I see such tasks as possibly being made available through teachers guides that might support curriculum documents, but also being added to by teachers in staff discussions and modified as a result of teachers experience with them. I see the taught curriculum as emerging as teachers occasion such activities and use official curriculum documents as checklists to see that opportunities have been provided for student to develop within the various aspects of knowing, doing, thinking, and the fostering of the more general aims of education. I envisage such a set of tasks or activities as being dynamic as not all will be suitable for all students and new tasks will continually be developed.

From this perspective the curriculum will become activity-based and curriculum documents would be used as checklists rather than as the basis for planning. This would shift curriculum responsibility from educational authorities back to teachers. This fits with the notion of teacher empowerment which is an important factor in teachers professional development (Robinson, 1989). However, to achieve this change in focus might require considerable professional development but it could happen in stages as teachers gradually broadened their focus from curriculum to include processes, thinking, the aims of education, and so on. I would prefer such a development to be encouraged by exploratory studies by groups of teachers rather than by legislation.

**Assessment**

The major impediment to an activity-based curriculum with curriculum documents being used as checklists rather than for planning is the assessment industry. This industry seems to have experienced considerable difficulty in moving beyond
traditional behavioural assessment to assess both knowing and doing. How thinking and the more general aims of education might also be assessed, and in which subjects the various aspects of thinking and the aims should be assessed are likely to be problematic issues. I see good assessment activities ideally reflecting good teaching/learning activities but they are generally more complex than the assessment industry wants.

Perhaps we need to recognize the need not only for a shift in curriculum thinking, but also a shift with respect to assessment. I wonder if more emphasis should be given to in-school assessment and to students’ self assessment rather than a focus on accountability, standards, qualifications, and managerialism. We need to consider the educational purposes of assessment and how it might contribute positively to learning.

**Conclusion**

What the word curriculum means continues to evolve and it now seems appropriate to broaden the debate about possibilities and alternative ways of working with curriculum for the 21st century. In presenting this paper my intention is to contribute to such a debate by offering alternatives rather than solutions with the hope that this might stimulate further debate.

**References**


Munro R (Chair) (1969) *Education in change, Report of the curriculum review group, New Zealand Post-Primary Teachers’ Association*, Auckland: Longman Paul


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